

On the parabolic regime of a hyperbolic equation with weak dissipation: the coercive case

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Abstract

We consider a family of Kirchhoff equations with a small parameter ε in front of the second-order time-derivative, and a dissipation term with a coefficient which tends to 0 as $t \rightarrow +\infty$.

It is well-known that, when the decay of the coefficient is slow enough, solutions behave as solutions of the corresponding parabolic equation, and in particular they decay to 0 as $t \rightarrow +\infty$.

In this paper we consider the nondegenerate and coercive case, and we prove *optimal* decay estimates for the hyperbolic problem, and optimal decay-error estimates for the difference between solutions of the hyperbolic and the parabolic problem. These estimates show a quite surprising fact: in the coercive case the analogy between parabolic equations and dissipative hyperbolic equations is weaker than in the noncoercive case.

This is actually a result for the corresponding linear equations with time-dependent coefficients. The nonlinear term comes into play only in the last step of the proof.

Mathematics Subject Classification 2010 (MSC2010): 35B25, 35L72, 35B40.

Key words: hyperbolic-parabolic singular perturbation, quasilinear hyperbolic equations, nondegenerate hyperbolic equations, Kirchhoff equations, decay-error estimates, linear equations with time-dependent coefficients.

1 Introduction

Let H be a real Hilbert space. For every x and y in H , $|x|$ denotes the norm of x , and $\langle x, y \rangle$ denotes the scalar product of x and y . Let A be a self-adjoint linear operator on H with dense domain $D(A)$. We assume that A is nonnegative, namely $\langle Ax, x \rangle \geq 0$ for every $x \in D(A)$, so that for every $\alpha \geq 0$ the power $A^\alpha x$ is defined provided that x lies in a suitable domain $D(A^\alpha)$.

We consider the Cauchy problem

$$\varepsilon u_\varepsilon''(t) + \frac{1}{(1+t)^p} u_\varepsilon'(t) + m(|A^{1/2} u_\varepsilon(t)|^2) A u_\varepsilon(t) = 0 \quad \forall t \geq 0, \quad (1.1)$$

$$u_\varepsilon(0) = u_0, \quad u_\varepsilon'(0) = u_1, \quad (1.2)$$

where $\varepsilon > 0$ and $p \geq 0$ are real parameters, $m : [0, +\infty) \rightarrow (0, +\infty)$ is a locally Lipschitz continuous function, and $(u_0, u_1) \in D(A) \times D(A^{1/2})$.

The singular perturbation problem in its generality consists in proving the convergence of solutions of (1.1), (1.2) to solutions of the first order problem

$$\frac{1}{(1+t)^p} u'(t) + m(|A^{1/2} u(t)|^2) A u(t) = 0 \quad \forall t \geq 0, \quad (1.3)$$

$$u(0) = u_0, \quad (1.4)$$

obtained setting formally $\varepsilon = 0$ in (1.1), and omitting the second initial condition in (1.2).

Several cases have been considered in the last 30 years, depending on the nonlinearity (degenerate or nondegenerate), on the dissipative term (constant dissipation $p = 0$ or weak dissipation $p > 0$), and on the operator A (coercive or noncoercive). The main research lines concern global existence for the parabolic and the hyperbolic problem (at least when ε is small enough), decay estimates on $u(t)$, $u_\varepsilon(t)$, and $u_\varepsilon(t) - u(t)$ as $t \rightarrow +\infty$, error estimates on the difference as $\varepsilon \rightarrow 0^+$, and decay-error estimates, namely estimates describing in the same time the behavior of the difference $u_\varepsilon(t) - u(t)$ as $t \rightarrow +\infty$ and $\varepsilon \rightarrow 0^+$. The interested reader is referred to the survey [6], or to the more recent papers [2, 7, 8].

In this paper we focus on the case where the equation is *nondegenerate*, namely

$$\inf \{m(\sigma) : \sigma \geq 0\} =: \mu > 0, \quad (1.5)$$

and the operator is *coercive*, namely

$$\inf \{\langle Au, u \rangle : u \in D(A), |u| = 1\} =: \nu > 0. \quad (1.6)$$

Concerning the parabolic problem, it is well-known that it admits a global solution for every $p \geq 0$, and every $u_0 \in D(A)$ (and even for less regular data and more general nonlinearities, see [9]).

As for the hyperbolic problem, things are different depending on p . Let us begin with the linear equation in which $m(\sigma)$ is a positive constant. In this case, T. Yamazaki [14] and J. Wirth [13] proved two complementary results, which can be outlined as follows.

- When $p > 1$, the dissipative term is too weak, and solutions of (1.1), (1.2) behave as solutions of the same equation without the dissipative term. In particular, solutions do not decay to 0. This is the *hyperbolic regime*.
- When $p < 1$, inertia is negligible, and solutions of (1.1), (1.2) behave as solutions of (1.3), (1.4). In particular, they decay to 0. This is the *parabolic regime*, with the so-called effective dissipation.
- When $p = 1$, the dissipation is still effective (namely the integral of the coefficient diverges), but according to [13] “the parabolic asymptotics changes to a wave type asymptotics”. In any case, solutions keep on going to 0, at least when ε is small enough, and for this reason the case $p = 1$ eventually falls in the parabolic regime.

These results have been extended to Kirchhoff equation by H. Hashimoto and T. Yamazaki [10], T. Yamazaki [15, 16] and the authors [5], in the following sense.

- When $p \in [0, 1]$, problem (1.1), (1.2) has a unique global solution provided that ε is small enough, and this solution decays to 0 as $t \rightarrow +\infty$. This is the parabolic regime.
- When $p > 1$, existence of global solutions to (1.1), (1.2) is known only for special initial data or special operators, the same ones for which global existence is known in the nondissipative case. Global existence for every $(u_0, u_1) \in D(A) \times D(A^{1/2})$, even for ε small enough, is still an open problem, exactly as in the nondissipative case. In any case, nontrivial global solutions, if they exist, can *not* decay to 0 as $t \rightarrow +\infty$. This is the hyperbolic regime.

All the results quoted above do not depend on the coerciveness of A , namely they are true also when $\nu = 0$.

Several estimates on solutions have been proved in the literature, once again without assumption (1.6). The prototype of *decay estimates* is that

$$|A^{1/2}u(t)|^2 \leq \frac{C}{(1+t)^{1+p}} \quad \text{and} \quad |A^{1/2}u_\varepsilon(t)|^2 \leq \frac{C}{(1+t)^{1+p}}$$

for every $t \geq 0$, where the constant C is independent of ε and of course also of t . As a consequence, we have also that

$$|A^{1/2}(u_\varepsilon(t) - u(t))|^2 \leq \frac{C}{(1+t)^{1+p}} \quad \forall t \geq 0. \quad (1.7)$$

The prototype of *error estimates* is that for initial data $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$ one has that

$$|A^{1/2}(u_\varepsilon(t) - u(t))|^2 \leq C\varepsilon^2 \quad \forall t \geq 0, \quad (1.8)$$

where the constant C is once again independent of ε and t (global-in-time error estimates). It is well-known that ε^2 is the best possible convergence rate (even when looking for local-in-time error estimates), and that $D(A^{3/2}) \times D(A^{1/2})$ is the minimal requirement on initial data which guarantees this rate (even in the case of linear equations). We refer to [1, 3, 4] for these aspects.

The prototype of *decay-error estimates* is that for initial data $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$ one has that

$$|A^{1/2}(u_\varepsilon(t) - u(t))|^2 \leq C \frac{\varepsilon^2}{(1+t)^{1+p}} \quad \forall t \geq 0. \quad (1.9)$$

We point out in particular that, according to these estimates, solutions of the hyperbolic problem decay with the same rate of solutions of the parabolic problem. Moreover, in the decay-error estimates (1.9) we have the same convergence rate of the error estimates (1.8), and the same decay rate of the decay estimates (1.7). Finally, all these results hold true without coerciveness assumptions on A , and for these general operators it turns out that decay rates are optimal.

When the operator A is coercive, better decay rates are expected. For example, it is easy to see that solutions of the parabolic problem satisfy

$$|A^{1/2}u(t)|^2 \leq Ce^{-\alpha(1+t)^{1+p}} \quad \forall t \geq 0 \quad (1.10)$$

for a suitable $\alpha > 0$, depending on μ , ν , and p (see Theorem 2.1).

Therefore, the analogy with the noncoercive case could lead to guess that also solutions of the hyperbolic problem should decay with the same exponential rate, and the same rate should also appear in the decay-error estimates.

In this paper we show that this is *not* the case, because solutions of the hyperbolic problem decay to 0 with a different, slower rate. Indeed we prove (see Theorem 2.2) that

$$|A^{1/2}u_\varepsilon(t)|^2 \leq Ce^{-\alpha(1+t)^{1-p}} \quad \forall t \geq 0 \quad (1.11)$$

if $p \in [0, 1)$, and

$$|A^{1/2}u_\varepsilon(t)|^2 \leq \frac{C}{(1+t)^\alpha} \quad \forall t \geq 0$$

if $p = 1$, where $\alpha < 2\mu\nu$ if $p = 0$, and α is any (positive) real number if $p \in (0, 1]$ (now the constant C depends also on α). These rates are optimal, in the sense that every nonzero solution does not satisfy an estimate such as (1.11) with an exponent larger than $(1-p)$ (see Theorem 2.4). The same slower rates appear also in the decay-error estimates (see Theorem 2.3), and of course they are optimal also in this case.

We have thus shown an essential difference between the coercive and the noncoercive case. In the noncoercive case, solutions of the hyperbolic problem mimic the behavior of solutions of the parabolic problem for every $p \in [0, 1]$. In the coercive case, this is true only for $p = 0$, when the exponent $(1 + p)$ in (1.10) and the exponent $(1 - p)$ in (1.11) coincide. On the contrary, for every $p \in (0, 1]$ there is a spread between exponents in the decay rates of $u(t)$ and $u_\varepsilon(t)$, and this spread becomes larger and larger as p approaches 1. As a consequence, from the point of view of decay rates, (1.3) is a good approximation of (1.1) for ε small in the noncoercive case, but not in the coercive case (see also section 2.3).

In both cases (coercive and noncoercive), the parabolic problem and the hyperbolic problem take different paths when $p > 1$: solutions of the parabolic problem keep on decaying according to (1.10), hence faster and faster as p grows, while solutions of the hyperbolic problem do not decay to 0 any more (provided that they globally exist).

All our proofs are based on linear arguments. To this end, we first linearize (1.1) and (1.3). We obtain the following equations

$$\varepsilon u_\varepsilon''(t) + \frac{1}{(1+t)^p} u_\varepsilon'(t) + c_\varepsilon(t) A u_\varepsilon(t) = 0 \quad \forall t \geq 0, \quad (1.12)$$

$$\frac{1}{(1+t)^p} u'(t) + c(t) A u(t) = 0 \quad \forall t \geq 0, \quad (1.13)$$

with time-dependent coefficients $c_\varepsilon : [0, +\infty) \rightarrow (0, +\infty)$ and $c : [0, +\infty) \rightarrow (0, +\infty)$.

Then we prove decay and decay-error estimates for solutions of these linear equations, under suitable assumptions on the coefficients. This is the core of the paper.

Finally, we just observe that the coefficients $c_\varepsilon(t)$ and $c(t)$ coming from the nonlinear terms in (1.1) and (1.3) satisfy the assumptions required by the linear theory. Fortunately, these assumptions are quite weak, and follow easily from previous literature on the noncoercive case.

This paper is organized as follows. In section 2.1 we recall the previous results and estimates needed throughout this paper. In section 2.2 we state our main results for Kirchhoff equations. In section 2.3 we present a heuristic argument leading to our decay rates. In section 2.4 we state our results for linear equations with time-dependent coefficients. In section 3 we collect all proofs.

2 Statements

2.1 Previous works

The theory of nondegenerate Kirchhoff equations with weak dissipation has been developed in [15, 16, 5]. In the following statement we collect the existence results, and some decay and error estimates. We limit ourselves to the results which are needed in the sequel, and for this reason Theorem A below does not represent the full state of the art,

especially for decay-error estimates. The interested reader is referred to section 5 of [6] for further (and more refined) estimates and references.

Theorem A *Let H be a Hilbert space, let A be a self-adjoint nonnegative operator on H with dense domain $D(A)$ (no coercivity assumption on A), let $m : [0, +\infty) \rightarrow (0, +\infty)$ be a locally Lipschitz continuous function satisfying the nondegeneracy condition (1.5), and let $(u_0, u_1) \in D(A) \times D(A^{1/2})$.*

Then we have the following conclusions.

- (1) (Parabolic problem) *For every $p \geq 0$, problem (1.3), (1.4) has a unique global solution*

$$u \in C^1([0, +\infty); H) \cap C^0([0, +\infty); D(A)). \quad (2.1)$$

Moreover $u \in C^1((0, +\infty); D(A^\alpha))$ for every $\alpha \geq 0$ (and more generally u is of class C^{k+1} when $m(\sigma)$ is of class C^k), and there exists a constant C such that

$$(1+t)^2 |u'(t)|^2 + (1+t)^{1+p} |A^{1/2}u(t)|^2 + (1+t)^{2(1+p)} |Au(t)|^2 \leq C \quad \forall t \geq 0. \quad (2.2)$$

- (2) (Hyperbolic problem) *For every $p \in [0, 1]$, there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, problem (1.1), (1.2) has a unique global solution*

$$u_\varepsilon \in C^2([0, +\infty); H) \cap C^1([0, +\infty); D(A^{1/2})) \cap C^0([0, +\infty); D(A)). \quad (2.3)$$

Moreover, there exists a constant C such that for every $\varepsilon \in (0, \varepsilon_0)$ we have that

$$(1+t)^2 |u'_\varepsilon(t)|^2 + (1+t)^{1+p} |A^{1/2}u_\varepsilon(t)|^2 + (1+t)^{2(1+p)} |Au_\varepsilon(t)|^2 \leq C \quad \forall t \geq 0. \quad (2.4)$$

- (3) (Singular perturbation) *If $p \in [0, 1]$, and $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$, then there exist $\varepsilon_1 \in (0, \varepsilon_0)$ and C such that, for every $\varepsilon \in (0, \varepsilon_1)$ we have that*

$$|A^{1/2}(u_\varepsilon(t) - u(t))|^2 \leq C\varepsilon^2 \quad \forall t \geq 0. \quad (2.5)$$

2.2 Main results

In this section we state the main results of this paper. The first one concerns decay estimates for solutions of the parabolic problem.

Theorem 2.1 (Parabolic equation) *Let H be a Hilbert space, and let A be a self-adjoint operator on H with dense domain $D(A)$. Let $u_0 \in D(A)$, let $p \geq 0$, and let $m : [0, +\infty) \rightarrow (0, +\infty)$ be a locally Lipschitz continuous function.*

Let us assume that the nondegeneracy and coerciveness assumptions (1.5) and (1.6) are satisfied.

Then problem (1.3), (1.4) has a unique global solution $u(t)$ with the regularity prescribed in statement (1) of Theorem A, and there exists a constant C such that

$$|u(t)|^2 + |A^{1/2}u(t)|^2 + |Au(t)|^2 + \frac{|u'(t)|^2}{(1+t)^{2p}} \leq C \exp\left(-\frac{2\mu\nu}{1+p}(1+t)^{1+p}\right) \quad (2.6)$$

for every $t \geq 0$.

The second result concerns decay estimates for solutions of the hyperbolic problem.

Theorem 2.2 (Hyperbolic equation) *Let H be a Hilbert space, and let A be a self-adjoint operator on H with dense domain $D(A)$. Let $(u_0, u_1) \in D(A) \times D(A^{1/2})$, let $p \in [0, 1]$, and let $m : [0, +\infty) \rightarrow (0, +\infty)$ be a locally Lipschitz continuous function.*

Let us assume that the nondegeneracy and coerciveness assumptions (1.5) and (1.6) are satisfied.

Then there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, problem (1.1), (1.2) has a unique global solution $u_\varepsilon(t)$ with the regularity prescribed by (2.3).

Moreover the function

$$\Gamma_\varepsilon(t) := |u_\varepsilon(t)|^2 + |A^{1/2}u_\varepsilon(t)|^2 + |Au_\varepsilon(t)|^2 + |u'_\varepsilon(t)|^2 + \varepsilon|A^{1/2}u'_\varepsilon(t)|^2 \quad (2.7)$$

satisfies the following decay estimates.

- Case $p = 0$ *For every $\beta < 2\mu\nu$, there exist $\varepsilon_1 \in (0, \varepsilon_0]$ and C such that*

$$\Gamma_\varepsilon(t) \leq Ce^{-\beta t} \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_1). \quad (2.8)$$

- Case $p \in (0, 1)$ *For every $\beta > 0$, there exist $\varepsilon_1 \in (0, \varepsilon_0]$ and C such that*

$$\Gamma_\varepsilon(t) \leq Ce^{-\beta(1+t)^{1-p}} \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_1). \quad (2.9)$$

- Case $p = 1$ *For every $\beta > 0$, there exist $\varepsilon_1 \in (0, \varepsilon_0]$ and C such that*

$$\Gamma_\varepsilon(t) \leq \frac{C}{(1+t)^\beta} \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_1). \quad (2.10)$$

Of course the constants C and ε_1 in (2.8) through (2.10) depend also on β .

The third step concerns the singular perturbation problem. Following the approach introduced in [11] in the linear case, we define the corrector $\theta_\varepsilon(t)$ as the solution of the second order *linear* ordinary differential equation

$$\varepsilon\theta''_\varepsilon(t) + \frac{1}{(1+t)^p}\theta'_\varepsilon(t) = 0 \quad \forall t \geq 0, \quad (2.11)$$

with initial data

$$\theta_\varepsilon(0) = 0, \quad \theta'_\varepsilon(0) = u_1 + m(|A^{1/2}u_0|^2) Au_0 =: \theta_0.$$

Since $\theta_0 = u'_\varepsilon(0) - u'(0)$, this corrector keeps into account the boundary layer due to the loss of one initial condition.

We can now define $r_\varepsilon(t)$ and $\rho_\varepsilon(t)$ in such a way that

$$u_\varepsilon(t) = u(t) + \theta_\varepsilon(t) + r_\varepsilon(t) = u(t) + \rho_\varepsilon(t) \quad \forall t \geq 0.$$

With these notations, the singular perturbation problem consists in proving that $r_\varepsilon(t) \rightarrow 0$ or $\rho_\varepsilon(t) \rightarrow 0$ in some sense as $\varepsilon \rightarrow 0^+$. We recall that the two remainders play different roles. In particular, $r_\varepsilon(t)$ is well suited for estimating derivatives, while $\rho_\varepsilon(t)$ is used in estimates without derivatives. This distinction is essential. Indeed it is not possible to prove decay-error estimates on $A^\alpha r_\varepsilon(t)$ because it does not decay to 0 as $t \rightarrow +\infty$ (indeed $u_\varepsilon(t)$ and $u(t)$ tend to 0, while the corrector $\theta_\varepsilon(t)$ does not), and it is not possible to prove decay-error estimates on $A^\alpha \rho'_\varepsilon(t)$ because in general for $t = 0$ it does not tend to 0 as $\varepsilon \rightarrow 0^+$ (due to the loss of one initial condition).

We are now ready to state our decay-error estimates.

Theorem 2.3 (Singular perturbation) *Let H be a Hilbert space, and let A be a self-adjoint operator on H with dense domain $D(A)$. Let $(u_0, u_1) \in D(A) \times D(A^{1/2})$, let $p \in [0, 1]$, and let $m : [0, +\infty) \rightarrow (0, +\infty)$ be a locally Lipschitz continuous function.*

Let us assume that the nondegeneracy and coerciveness assumptions (1.5) and (1.6) are satisfied, and let $u(t)$, ε_0 , $u_\varepsilon(t)$, $r_\varepsilon(t)$, $\rho_\varepsilon(t)$ be as above.

Let us consider the functions

$$\Gamma_{r,\varepsilon}(t) := |\rho_\varepsilon(t)|^2 + |A^{1/2}\rho_\varepsilon(t)|^2 + \varepsilon|r'_\varepsilon(t)|^2,$$

$$\Gamma_{c,\varepsilon}(t) := |\rho_\varepsilon(t)|^2 + |A^{1/2}\rho_\varepsilon(t)|^2 + |A\rho_\varepsilon(t)|^2 + |r'_\varepsilon(t)|^2 + \varepsilon|A^{1/2}r'_\varepsilon(t)|^2,$$

where indices c and r stay for “complete”, and “reduced”, respectively.

(1) *If in addition $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$, then we have the following decay-error estimates.*

- Case $p = 0$ *For every $\beta < 2\mu\nu$, there exist $\varepsilon_1 \in (0, \varepsilon_0]$ and C such that*

$$\Gamma_{r,\varepsilon}(t) \leq C\varepsilon^2 e^{-\beta t} \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_1). \quad (2.12)$$

- Case $p \in (0, 1)$ *For every $\beta > 0$, there exist $\varepsilon_1 \in (0, \varepsilon_0]$ and C such that*

$$\Gamma_{r,\varepsilon}(t) \leq C\varepsilon^2 e^{-\beta(1+t)^{1-p}} \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_1). \quad (2.13)$$

- Case $p = 1$ *For every $\beta > 0$, there exist $\varepsilon_1 \in (0, \varepsilon_0]$ and C such that*

$$\Gamma_{r,\varepsilon}(t) \leq \frac{C\varepsilon^2}{(1+t)^\beta} \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_1). \quad (2.14)$$

(2) *If in addition $(u_0, u_1) \in D(A^2) \times D(A)$, then we have the same decay-error estimates with $\Gamma_{c,\varepsilon}(t)$ instead of $\Gamma_{r,\varepsilon}(t)$.*

As in Theorem 2.2 above, the constants C and ε_1 in (2.12) through (2.14) depend also on β . We point out that in these estimates we have the same convergence rate as in (2.5), and the same decay rates as in (2.8) through (2.10).

The last result we state, together with Remarks 2.5 and 2.6 below, clarifies the optimality of the decay rates of Theorem 2.2, hence also of Theorem 2.3.

Theorem 2.4 (Optimality of decay rates) *Let $H, A, p \in [0, 1]$, $m : [0, +\infty) \rightarrow (0, +\infty)$, and $(u_0, u_1) \in D(A) \times D(A^{1/2})$ be as in Theorem 2.2. Let $\varepsilon > 0$, and let $u_\varepsilon(t)$ be the solution to problem (1.1), (1.2).*

Let $\Phi : [0, +\infty) \rightarrow (0, +\infty)$ be a function of class C^1 such that

$$\lim_{t \rightarrow +\infty} (1+t)^p \frac{\Phi'(t)}{\Phi(t)} = -\infty. \quad (2.15)$$

If $(u_0, u_1) \neq (0, 0)$, then

$$\lim_{t \rightarrow +\infty} (\varepsilon |u'_\varepsilon(t)|^2 + |A^{1/2} u_\varepsilon(t)|^2) \frac{1}{\Phi(t)} = +\infty. \quad (2.16)$$

Remark 2.5 When $p > 0$, Theorem 2.4 is exactly the counterpart of Theorem 2.2. Indeed let us consider any $\Phi : [0, +\infty) \rightarrow [0, +\infty)$, and let $\Gamma_\varepsilon(t)$ be defined as in (2.7). If (2.15) is satisfied, then we can not expect that $\Gamma_\varepsilon(t) \leq C\Phi(t)$ because of (2.16). On the contrary, if

$$(1+t)^p \frac{\Phi'(t)}{\Phi(t)} \geq -\beta > -\infty,$$

then $\Phi(t) \geq Ce^{-\beta(1-p)^{-1}(1+t)^{1-p}}$ if $p \in (0, 1)$, and $\Phi(t) \geq C(1+t)^{-\beta}$ if $p = 1$, and in both cases Theorem 2.2 guarantees that $\Gamma_\varepsilon(t) \leq C\Phi(t)$.

Note in particular that the function $\Phi(t) := e^{-\beta(1+t)^\delta}$ satisfies (2.15) if and only if $\delta > 1 - p$, which means that $(1 - p)$ is the larger exponent for which (2.9) holds true.

Remark 2.6 When $p = 0$, estimate (2.8) can not be true when $\beta > 2\mu\nu$. This can be easily seen by considering the explicit solutions of the ordinary differential equation

$$\varepsilon y''(t) + y'(t) + \mu\nu y(t) = 0, \quad (2.17)$$

which is just the particular case of (1.1) where $H = \mathbb{R}$, A is ν times the identity, and $m(\sigma) \equiv \mu$ is a constant.

On the other hand, solutions of (2.17) satisfy (2.8) also with $\beta = 2\mu\nu$. We suspect that this could be true in general, but for the time being we have no proof.

Open problem 2.7 Is (2.8) true also in the case $\beta = 2\mu\nu$?

2.3 Heuristics

According to Theorem 2.1, solutions of the parabolic problem decay as solutions of the ordinary differential equation

$$\frac{1}{(1+t)^p} y'(t) + \mu\nu y(t) = 0. \quad (2.18)$$

This is hardly surprising, since (2.18) is just the special case of (1.3) corresponding to $H = \mathbb{R}$, A equal to ν times the identity, and $m(\sigma) \equiv \mu$.

Analogously, it is reasonable to expect solutions of the hyperbolic problem to decay as solutions of the ordinary differential equation

$$\varepsilon y_\varepsilon''(t) + \frac{1}{(1+t)^p} y_\varepsilon'(t) + \mu\nu y_\varepsilon(t) = 0. \quad (2.19)$$

A reasonable ansatz for these solutions is that asymptotically they are the product of an oscillatory term $v_\varepsilon(t)$, and a decaying term $\lambda_\varepsilon(t)$. Plugging $y_\varepsilon(t) = \lambda_\varepsilon(t) \cdot v_\varepsilon(t)$ into (2.19), we obtain that

$$(\varepsilon v_\varepsilon''(t) + \mu\nu v_\varepsilon(t)) \lambda_\varepsilon(t) + \left(2\varepsilon \lambda_\varepsilon'(t) + \frac{\lambda_\varepsilon(t)}{(1+t)^p}\right) v_\varepsilon'(t) + \left(\varepsilon \lambda_\varepsilon''(t) + \frac{\lambda_\varepsilon'(t)}{(1+t)^p}\right) v_\varepsilon(t) = 0.$$

A reasonable guess is now that the coefficient of $\lambda_\varepsilon(t)$ in the first term is almost 0, as well as the coefficient of $v_\varepsilon'(t)$ in the second term.

The first condition is that $\varepsilon v_\varepsilon''(t) + \mu\nu v_\varepsilon(t) \sim 0$, namely

$$v_\varepsilon(t) \sim \sin\left(\sqrt{\frac{\mu\nu}{\varepsilon}}t\right),$$

which yields the same oscillations of the undamped equation.

The second condition is that

$$2\varepsilon \lambda_\varepsilon'(t) + \frac{\lambda_\varepsilon(t)}{(1+t)^p} \sim 0, \quad (2.20)$$

and for every $p \in (0, 1]$ this yields a decay rate which is compatible both with Theorem 2.2 and with Theorem 2.4.

We do not know if similar asymptotics have been rigorously justified in the literature (see [13] for the case $p = 1$). Nevertheless, this non-rigorous argument suggests that actually there is no sharp break between parabolic and hyperbolic regimes. For $p \leq 1$, the hyperbolic nature survives in the oscillatory behavior of $v_\varepsilon(t)$, but it is hidden by the damping imposed by (2.20). When $p > 1$, solutions of (2.20) tend to a positive constant, and the hyperbolic nature emerges undisputed.

We conclude by pointing out once again that this analysis applies to the nondegenerate coercive case. Things are quite different both in the nondegenerate noncoercive case (see [5, 12, 13, 14, 15]), and in the degenerate coercive case (see [7, 8]).

2.4 Linearization

Proofs of our main results are based on the analysis of the linear equations (1.12) and (1.13). We assume that the coefficient $c : [0, +\infty) \rightarrow (0, +\infty)$ is of class C^1 , and satisfies the following estimates

$$c(t) \geq \mu > 0 \quad \forall t \geq 0, \quad (2.21)$$

$$c(t) \leq M_1 \quad \forall t \geq 0, \quad (2.22)$$

$$|c'(t)| \leq M_2 \quad \forall t \geq 0. \quad (2.23)$$

Similarly, we assume that $c_\varepsilon : [0, +\infty) \rightarrow (0, +\infty)$, with $\varepsilon \in (0, \varepsilon_0)$, is a family of coefficients of class C^1 satisfying the following estimates

$$c_\varepsilon(t) \geq \mu > 0 \quad \forall t \geq 0, \quad \forall \varepsilon \in (0, \varepsilon_0), \quad (2.24)$$

$$c_\varepsilon(t) \leq M_3 \quad \forall t \geq 0, \quad \forall \varepsilon \in (0, \varepsilon_0), \quad (2.25)$$

$$|c'_\varepsilon(t)| \leq \frac{M_4}{(1+t)^p} \quad \forall t \geq 0, \quad \forall \varepsilon \in (0, \varepsilon_0). \quad (2.26)$$

When considering the singular perturbation, we also assume that

$$|c_\varepsilon(t) - c(t)| \leq M_5 \varepsilon \quad \forall t \geq 0, \quad \forall \varepsilon \in (0, \varepsilon_0), \quad (2.27)$$

and we define the corrector $\theta_\varepsilon(t)$ as the solution of (2.11) with initial data

$$\theta_\varepsilon(0) = 0, \quad \theta'_\varepsilon(0) = u_1 + c(0)Au_0 =: \theta_0. \quad (2.28)$$

The following results are the linear counterparts of Theorems 2.1 through 2.4. All of them can be extended to Lipschitz continuous coefficients through a straightforward approximation argument.

Theorem 2.8 (Linear parabolic equation) *Let H , A , $p \geq 0$, and $u_0 \in D(A)$ be as in Theorem 2.1. Let $c : [0, +\infty) \rightarrow (0, +\infty)$ be a continuous function satisfying (2.21) and (2.22).*

Then problem (1.13), (1.4) has a unique global solution $u(t)$ with the regularity prescribed by (2.1), and this solution satisfies (2.6).

Theorem 2.9 (Linear hyperbolic equation) *Let H , A , $p \in [0, 1]$, and $(u_0, u_1) \in D(A) \times D(A^{1/2})$ be as in Theorem 2.2, and let $\varepsilon_0 > 0$. Let $c_\varepsilon : [0, +\infty) \rightarrow (0, +\infty)$, with $\varepsilon \in (0, \varepsilon_0)$, be a family of coefficients of class C^1 satisfying (2.24) through (2.26).*

Then, for every $\varepsilon \in (0, \varepsilon_0)$, problem (1.12), (1.2) has a unique global solution $u_\varepsilon(t)$ with the regularity prescribed by (2.3), and this solution satisfies the same decay estimates stated in Theorem 2.2, depending on the values of p .

Theorem 2.10 (Linear singular perturbation) *Let H , A , $p \in [0, 1]$, (u_0, u_1) , ε_0 , $c(t)$, $u(t)$, $c_\varepsilon(t)$, $u_\varepsilon(t)$ be as in Theorems 2.8 and 2.9.*

Let us assume that also (2.23) and (2.27) hold true, and let $r_\varepsilon(t)$ and $\rho_\varepsilon(t)$ be defined as usual (keeping in mind that the corrector now satisfies (2.11) and (2.28)).

Then $r_\varepsilon(t)$ and $\rho_\varepsilon(t)$ satisfy the decay-error estimates of statements (1) and (2) of Theorem 2.3, depending on the further regularity of (u_0, u_1) , and on the values of p .

Theorem 2.11 (Linear hyperbolic equation: optimality) *Let $H, A, p \in [0, 1]$, and $(u_0, u_1) \in D(A) \times D(A^{1/2})$ be as in Theorem 2.2. Let $\varepsilon > 0$, and let $u_\varepsilon(t)$ be the solution to problem (1.12), (1.2) with a coefficient $c_\varepsilon : [0, +\infty) \rightarrow (0, +\infty)$ of class C^1 satisfying (2.24) through (2.26).*

If $(u_0, u_1) \neq (0, 0)$, then (2.16) holds true for every function $\Phi : [0, +\infty) \rightarrow (0, +\infty)$ of class C^1 satisfying (2.15).

3 Proofs

3.1 Proof of Theorem 2.8

We prove a more general result, with some further estimates needed when dealing with the singular perturbation problem.

Proposition 3.1 *Let H, A , and $c(t)$ be as in Theorem 2.8. Let us set*

$$\gamma := \frac{2\mu\nu}{1+p}, \quad \Psi_{\alpha,p}(t) := \exp\left(-\alpha[(1+t)^{1+p} - 1]\right). \quad (3.1)$$

Then we have the following estimates.

(1) *If $u_0 \in D(A^{k/2})$ for some $k \in \mathbb{N}$, then*

$$|A^{k/2}u(t)|^2 \leq |A^{k/2}u_0|^2 \Psi_{\gamma,p}(t) \quad \forall t \geq 0. \quad (3.2)$$

Moreover, for every $\alpha < \gamma$ we have that

$$\int_0^{+\infty} \frac{|A^{(k+1)/2}u(t)|^2}{\Psi_{\alpha,p}(t)} dt \leq \left(2\mu - \frac{\alpha(1+p)}{\nu}\right)^{-1} |A^{k/2}u_0|^2. \quad (3.3)$$

(2) *If $u_0 \in D(A^{3/2})$, and $c(t)$ is of class C^1 and satisfies (2.23), then for every $\alpha < \gamma$ there exists a constant C (depending also on α) such that*

$$\int_0^{+\infty} \frac{|u''(t)|^2}{\Psi_{\alpha,p}(t)} dt \leq C. \quad (3.4)$$

(3) *If $u_0 \in D(A^2)$, and $c(t)$ is of class C^1 and satisfies (2.23), then there exists a constant C such that*

$$|u''(t)|^2 \leq C(1+t)^{4p} \Psi_{\gamma,p}(t) \quad \forall t \geq 0. \quad (3.5)$$

Moreover, for every $\alpha < \gamma$, there exists a constant C (depending also on α) such that

$$\int_0^{+\infty} \frac{|A^{1/2}u''(t)|^2}{\Psi_{\alpha,p}(t)} dt \leq C. \quad (3.6)$$

Proof Let us set $E_k(t) := |A^{k/2}u(t)|^2$. From (1.13), (1.6), and (2.21), we have that

$$\begin{aligned} E'_k(t) &= 2\langle A^{(k+1)/2}u(t), A^{(k-1)/2}u'(t) \rangle = -2c(t)(1+t)^p |A^{(k+1)/2}u(t)|^2 \\ &\leq -2c(t)(1+t)^p \cdot \nu |A^{k/2}u(t)|^2 \leq -2\mu\nu(1+t)^p E_k(t). \end{aligned}$$

Integrating this differential inequality, we obtain (3.2).

Moreover we have that

$$\begin{aligned} \frac{d}{dt} \left[\frac{E_k(t)}{\Psi_{\alpha,p}(t)} \right] &= \frac{E'_k(t)}{\Psi_{\alpha,p}(t)} + \alpha(1+p)(1+t)^p \frac{|A^{k/2}u(t)|^2}{\Psi_{\alpha,p}(t)} \\ &\leq -2\mu(1+t)^p \frac{|A^{(k+1)/2}u(t)|^2}{\Psi_{\alpha,p}(t)} + \frac{\alpha(1+p)}{\nu}(1+t)^p \frac{|A^{(k+1)/2}u(t)|^2}{\Psi_{\alpha,p}(t)}, \end{aligned}$$

hence

$$\left(2\mu - \frac{\alpha(1+p)}{\nu} \right) \int_0^t (1+s)^p \frac{|A^{(k+1)/2}u(s)|^2}{\Psi_{\alpha,p}(s)} ds + \frac{E_k(t)}{\Psi_{\alpha,p}(t)} \leq E_k(0) \quad \forall t \geq 0,$$

which easily implies (3.3).

Let us prove the estimates on the second derivative. From (1.13) we obtain that

$$u''(t) = -p(1+t)^{p-1}c(t)Au(t) - (1+t)^p c'(t)Au(t) + (1+t)^{2p}c^2(t)A^2u(t).$$

Therefore, from (2.22) and (2.23), it follows that

$$|u''(t)|^2 \leq k_1(1+t)^{2p}|Au(t)|^2 + k_2(1+t)^{4p}|A^2u(t)|^2. \quad (3.7)$$

If $u_0 \in D(A^2)$, then (3.5) follows from (3.2) with $k = 2$ and $k = 4$.

In order to prove the integral estimates on $u''(t)$, let us choose η such that $\alpha < \alpha + \eta < \gamma$. Since $\Psi_{\alpha+\eta,p}(t) = \Psi_{\alpha,p}(t) \cdot \Psi_{\eta,p}(t)$, and since

$$\sup_{t \geq 0} \{ \Psi_{\eta,p}(t)(1+t)^{4p} \} < +\infty,$$

from (3.7) it follows that

$$\frac{|u''(t)|^2}{\Psi_{\alpha,p}(t)} \leq (1+t)^{4p} \Psi_{\eta,p}(t) \cdot \frac{k_1|Au(t)|^2 + k_2|A^2u(t)|^2}{\Psi_{\alpha,p}(t) \cdot \Psi_{\eta,p}(t)} \leq k_3 \frac{|Au(t)|^2 + |A^2u(t)|^2}{\Psi_{\alpha+\eta,p}(t)}.$$

From (3.3) with $k = 1$ and $k = 3$ we conclude that

$$\int_0^{+\infty} \frac{|u''(t)|^2}{\Psi_{\alpha,p}(t)} dt \leq k_3 \int_0^{+\infty} \frac{|Au(t)|^2 + |A^2u(t)|^2}{\Psi_{\alpha+\eta,p}(t)} dt \leq k_4$$

for a suitable k_4 depending also on η . This proves (3.4).

The proof of (3.6) is completely analogous. \square

3.2 Comparison results for ODEs

In this subsection we prove estimates for solutions of three ordinary differential equations needed in the sequel. To begin with, for every $\beta > 0$ and every $p \geq 0$ we define $\Phi_{\beta,p} : [0, +\infty) \rightarrow (0, +\infty)$ as the solution of the Cauchy problem

$$\Phi'_{\beta,p}(t) = -\frac{\beta}{(1+t)^p} \Phi_{\beta,p}(t) \quad \forall t \geq 0, \quad (3.8)$$

$$\Phi_{\beta,p}(0) = 1. \quad (3.9)$$

We point out that solutions of this problem decay as the right-hand sides of (2.8) through (2.10), depending on the values of p . This is the reason why we are going to exploit $\Phi_{\beta,p}(t)$ several times in the proofs of our decay and decay-error estimates.

Lemma 3.2 *Let $\beta > 0$ and $p \geq 0$ be real numbers, and let $\Phi_{\beta,p}(t)$ be the solution of the Cauchy problem (3.8), (3.9).*

Let ε and K be positive constants, with $2\varepsilon\beta \leq 1$, and let $G : [0, +\infty) \rightarrow [0, +\infty)$ be a function of class C^1 such that

$$G'(t) \leq -\frac{1}{\varepsilon} \frac{1}{(1+t)^p} G(t) + \frac{K}{\varepsilon} (1+t)^p \Phi_{\beta,p}(t) \quad \forall t \geq 0. \quad (3.10)$$

Then we have that

$$G(t) \leq (2K + G(0)) (1+t)^{2p} \Phi_{\beta,p}(t) \quad \forall t \geq 0. \quad (3.11)$$

Proof Let us consider the differential equation

$$y'(t) = -\frac{1}{\varepsilon} \frac{1}{(1+t)^p} y(t) + \frac{K}{\varepsilon} (1+t)^p \Phi_{\beta,p}(t) \quad \forall t \geq 0. \quad (3.12)$$

Assumption (3.10) says that $G(t)$ is a subsolution of (3.12). Let $z(t)$ denote the right-hand side of (3.11). We claim that $z(t)$ is a supersolution of (3.12). Indeed a simple computation shows that

$$\begin{aligned} z'(t) &= 2p(2K + G(0))(1+t)^{2p-1} \Phi_{\beta,p}(t) + (2K + G(0))(1+t)^{2p} \Phi'_{\beta,p}(t) \\ &\geq -\beta(2K + G(0))(1+t)^p \Phi_{\beta,p}(t) \\ &\geq -\frac{1}{\varepsilon} (K + G(0))(1+t)^p \Phi_{\beta,p}(t) \\ &= -\frac{1}{\varepsilon} \frac{1}{(1+t)^p} z(t) + \frac{K}{\varepsilon} (1+t)^p \Phi_{\beta,p}(t), \end{aligned}$$

where in the second inequality we exploited that $2\varepsilon\beta \leq 1$, and $2G(0) \geq G(0)$.

Since $G(0) \leq z(0)$, estimate (3.11) follows from the standard comparison principle between subsolutions and supersolutions. \square

Lemma 3.3 Let $\psi_1 : [0, +\infty) \rightarrow [0, +\infty)$ and $\psi_2 : [0, +\infty) \rightarrow [0, +\infty)$ be two continuous functions such that

$$K_1 := \int_0^{+\infty} \psi_1(t) dt < +\infty, \quad K_2 := \int_0^{+\infty} \psi_2(t) dt < +\infty.$$

Let $E : [0, +\infty) \rightarrow [0, +\infty)$ be a function of class C^1 such that $E(0) = 0$, and

$$E'(t) \leq \psi_1(t) \sqrt{E(t)} + \psi_2(t) \quad \forall t \geq 0.$$

Then we have that

$$E(t) \leq K_1^2 + 2K_2 \quad \forall t \geq 0. \quad (3.13)$$

Proof Let us fix any $T > 0$. For every $t \in [0, T]$ we have that

$$E'(t) \leq \psi_1(t) \cdot \left(\sup_{s \in [0, T]} E(s) \right)^{1/2} + \psi_2(t).$$

Since $E(0) = 0$, an easy integration gives that

$$E(t) \leq \left(\sup_{s \in [0, T]} E(s) \right)^{1/2} \int_0^t \psi_1(s) ds + \int_0^t \psi_2(s) ds \leq K_1 \left(\sup_{s \in [0, T]} E(s) \right)^{1/2} + K_2$$

for every $t \in [0, T]$. Taking the supremum of the left-hand side as $t \in [0, T]$, we obtain that

$$\sup_{s \in [0, T]} E(s) \leq K_1 \left(\sup_{s \in [0, T]} E(s) \right)^{1/2} + K_2 \leq \frac{1}{2} K_1^2 + \frac{1}{2} \left(\sup_{s \in [0, T]} E(s) \right) + K_2,$$

hence

$$\sup_{s \in [0, T]} E(s) \leq K_1^2 + 2K_2,$$

and in particular $E(T) \leq K_1^2 + 2K_2$. Since T is arbitrary, (3.13) is proved. \square

Lemma 3.4 Let $\beta > 0$ and $p \geq 0$ be real numbers, and let $\Phi_{\beta, p}(t)$ be the solution of the Cauchy problem (3.8), (3.9).

Let $\psi : [0, +\infty) \rightarrow [0, +\infty)$ be a continuous function such that

$$\int_0^{+\infty} \frac{\psi(s)}{\Phi_{\beta, p}(s)} ds < +\infty.$$

Let $T > 0$, and let $F : [T, +\infty) \rightarrow [0, +\infty)$ be a function of class C^1 such that

$$F'(t) \leq -\frac{\beta}{(1+t)^p} F(t) + \psi(t) \quad \forall t \geq T. \quad (3.14)$$

Then we have that

$$F(t) \leq \left(\frac{F(T)}{\Phi_{\beta, p}(T)} + \int_0^{+\infty} \frac{\psi(s)}{\Phi_{\beta, p}(s)} ds \right) \cdot \Phi_{\beta, p}(t) \quad \forall t \geq T. \quad (3.15)$$

Proof Let us consider the differential equation

$$y'(t) = -\frac{\beta}{(1+t)^p}y(t) + \psi(t) \quad \forall t \geq 0. \quad (3.16)$$

Assumption (3.14) says that $F(t)$ is a subsolution of (3.16) for $t \geq T$. On the other hand, it is easy to see that

$$z(t) := \left(\frac{F(T)}{\Phi_{\beta,p}(T)} + \int_T^t \frac{\psi(s)}{\Phi_{\beta,p}(s)} ds \right) \cdot \Phi_{\beta,p}(t)$$

is a solution of (3.16) for $t \geq T$. Since $F(T) = z(T)$, the standard comparison principle between subsolutions and supersolutions implies that $F(t) \leq z(t)$ for every $t \geq T$, which in turn implies (3.15). \square

3.3 Proof of Theorem 2.9

Let us describe the strategy of the proof before entering into details. Let us take any admissible value β , which means any $\beta \in (0, 2\mu\nu)$ if $p = 0$, and any $\beta > 0$ if $p > 0$. Let $\Phi_{\beta,p}(t)$ be the solution of the Cauchy problem (3.8), (3.9).

Estimates (2.8) through (2.10) are equivalent to showing that

$$\Gamma_\varepsilon(t) \leq k_1 \Phi_{\beta,p}(t) \quad \forall t \geq 0 \quad (3.17)$$

for the admissible values of β .

Let μ be the constant in (2.24), and let us choose δ and T in such a way that

$$\delta := \frac{2(\beta+1)\nu}{2\mu\nu - \beta}, \quad T := 0 \quad (3.18)$$

if $p = 0$ (note that $\delta > 0$), and

$$\delta := \frac{\beta+2}{\mu}, \quad (1+T)^{2p} \geq \frac{\delta\beta}{2\nu} \quad (3.19)$$

if $p > 0$. For every $\varepsilon \in (0, \varepsilon_0)$, we consider the energies

$$E_\varepsilon(t) := \frac{\varepsilon |u'_\varepsilon(t)|^2}{c_\varepsilon(t)} + |A^{1/2}u_\varepsilon(t)|^2, \quad (3.20)$$

$$F_\varepsilon(t) := \frac{\varepsilon |u'_\varepsilon(t)|^2}{c_\varepsilon(t)} + |A^{1/2}u_\varepsilon(t)|^2 + \frac{\varepsilon\delta}{(1+t)^p} \langle u'_\varepsilon(t), u_\varepsilon(t) \rangle + \frac{\delta}{2} \frac{1}{(1+t)^{2p}} |u_\varepsilon(t)|^2. \quad (3.21)$$

We claim that there exist $\varepsilon_2 \in (0, \varepsilon_0)$, and positive constants k_2, \dots, k_5 , such that

$$k_2 (\varepsilon |u'_\varepsilon(t)|^2 + |A^{1/2}u_\varepsilon(t)|^2) \leq E_\varepsilon(t) \leq k_3 (\varepsilon |u'_\varepsilon(t)|^2 + |A^{1/2}u_\varepsilon(t)|^2), \quad (3.22)$$

$$k_4 (\varepsilon |u'_\varepsilon(t)|^2 + |A^{1/2}u_\varepsilon(t)|^2) \leq F_\varepsilon(t) \leq k_5 (\varepsilon |u'_\varepsilon(t)|^2 + |A^{1/2}u_\varepsilon(t)|^2) \quad (3.23)$$

for every $t \geq 0$ and every $\varepsilon \in (0, \varepsilon_2)$. Moreover we claim that

$$E'_\varepsilon(t) \leq 0 \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_2), \quad (3.24)$$

$$F'_\varepsilon(t) \leq -\frac{\beta}{(1+t)^p} F_\varepsilon(t) \quad \forall t \geq T, \forall \varepsilon \in (0, \varepsilon_2). \quad (3.25)$$

Let us assume that we have proved these claims. Thanks to (3.24), and to the estimate from below in (3.22), we have that

$$\varepsilon |u'_\varepsilon(t)|^2 + |A^{1/2}u_\varepsilon(t)|^2 \leq \frac{1}{k_2} E_\varepsilon(t) \leq \frac{1}{k_2} E_\varepsilon(0) \leq k_6$$

for every $t \geq 0$. Since $\Phi_{\beta,p}(t)$ is decreasing, this implies that

$$\varepsilon |u'_\varepsilon(t)|^2 + |A^{1/2}u_\varepsilon(t)|^2 \leq \frac{k_6}{\Phi_{\beta,p}(T)} \cdot \Phi_{\beta,p}(t) = k_7 \Phi_{\beta,p}(t) \quad \forall t \in [0, T]. \quad (3.26)$$

For $t \geq T$, we exploit (3.25). First of all, from (3.26) with $t = T$, and the estimate from above in (3.23), we have that

$$F_\varepsilon(T) \leq k_5 (\varepsilon |u'_\varepsilon(T)|^2 + |A^{1/2}u_\varepsilon(T)|^2) \leq k_8 \Phi_{\beta,p}(T).$$

Therefore, from Lemma 3.4 applied with $\psi(t) \equiv 0$, we deduce that $F_\varepsilon(t) \leq k_8 \Phi_{\beta,p}(t)$ for every $t \geq T$. Exploiting this inequality, the estimate from below in (3.23), and (3.26), we conclude that

$$\varepsilon |u'_\varepsilon(t)|^2 + |A^{1/2}u_\varepsilon(t)|^2 \leq k_9 \Phi_{\beta,p}(t) \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_2). \quad (3.27)$$

Since the operator is coercive, this estimate on $|A^{1/2}u_\varepsilon(t)|^2$ yields an analogous estimate on $|u_\varepsilon(t)|^2$.

Up to now, we only assumed that $(u_0, u_1) \in D(A^{1/2}) \times H$. Let us assume now that $(u_0, u_1) \in D(A) \times D(A^{1/2})$. Since equation (1.12) is linear, estimate (3.27) can be applied to $A^{1/2}u_\varepsilon(t)$, which is once again a solution to (1.12). We thus obtain that

$$\varepsilon |A^{1/2}u'_\varepsilon(t)|^2 + |Au_\varepsilon(t)|^2 \leq k_{10} \Phi_{\beta,p}(t) \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_2). \quad (3.28)$$

It remains to prove the ε -independent estimate on $|u'_\varepsilon(t)|^2$. To this end, we set

$$G_\varepsilon(t) := |u'_\varepsilon(t)|^2, \quad (3.29)$$

and we claim that

$$G'_\varepsilon(t) \leq -\frac{1}{\varepsilon} \frac{1}{(1+t)^p} G_\varepsilon(t) + \frac{k_{11}}{\varepsilon} (1+t)^p \Phi_{\beta,p}(t) \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_2). \quad (3.30)$$

If we prove the claim, then from Lemma 3.2 it follows that

$$|u'_\varepsilon(t)|^2 = G_\varepsilon(t) \leq k_{12}(1+t)^{2p}\Phi_{\beta,p}(t) \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_2). \quad (3.31)$$

What we actually need is the same estimate without the factor $(1+t)^{2p}$. If $p = 0$, there is nothing to do. If $p > 0$, we take $\beta' = \beta + 2$, and from (3.31) we obtain that

$$|u'_\varepsilon(t)|^2 = G_\varepsilon(t) \leq k_{13}(1+t)^{2p}\Phi_{\beta',p}(t) \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_1),$$

of course with new positive constants k_{13} and $\varepsilon_1 \leq \varepsilon_2$, depending also on β' .

Finally, our choice of β' guarantees that

$$(1+t)^{2p}\Phi_{\beta',p}(t) \leq k_{14}\Phi_{\beta,p}(t) \quad \forall t \geq 0$$

for a suitable k_{14} depending on p, β, β' (this inequality can be easily proved exploiting the explicit formulae for $\Phi_{\beta,p}(t)$ and $\Phi_{\beta',p}(t)$, and the fact that $p \leq 1$). This completes the proof of (3.17) for every $\varepsilon \in (0, \varepsilon_1)$.

So we are left to proving (3.22) through (3.25), and (3.30).

Equivalence between energies Due to (2.24) and (2.25), estimate (3.22) holds true with

$$k_2 := \min \left\{ \frac{1}{M_3}, 1 \right\}, \quad k_3 := \max \left\{ \frac{1}{\mu}, 1 \right\}.$$

In order to prove (3.23), let us estimate separately the four terms in (3.21). Due to (2.24) and (2.25), we have that

$$\frac{\varepsilon|u'_\varepsilon(t)|^2}{M_3} \leq \frac{\varepsilon|u'_\varepsilon(t)|^2}{c_\varepsilon(t)} \leq \frac{\varepsilon|u'_\varepsilon(t)|^2}{\mu}.$$

Due to (1.6) we have that

$$0 \leq \frac{\delta}{2} \frac{1}{(1+t)^{2p}} |u_\varepsilon(t)|^2 \leq \frac{\delta}{2} |u_\varepsilon(t)|^2 \leq \frac{\delta}{2\nu} |A^{1/2}u_\varepsilon(t)|^2.$$

Applying once again (1.6), and the inequality between arithmetic and geometric mean, we obtain that

$$\frac{\varepsilon|u'_\varepsilon(t)|^2}{2M_3} + \frac{1}{2} |A^{1/2}u_\varepsilon(t)|^2 \geq \frac{\varepsilon|u'_\varepsilon(t)|^2}{2M_3} + \frac{\nu}{2} |u_\varepsilon(t)|^2 \geq \sqrt{\frac{\varepsilon\nu}{M_3}} \cdot |u'_\varepsilon(t)| \cdot |u_\varepsilon(t)|.$$

If ε is small enough, this implies that

$$\frac{\varepsilon|u'_\varepsilon(t)|^2}{2M_3} + \frac{1}{2} |A^{1/2}u_\varepsilon(t)|^2 \geq \frac{\varepsilon\delta}{(1+t)^{2p}} |\langle u'_\varepsilon(t), u_\varepsilon(t) \rangle|.$$

From all these estimates, we easily obtain that

$$F_\varepsilon(t) \geq \frac{\varepsilon |u'_\varepsilon(t)|^2}{2M_3} + \frac{1}{2} |A^{1/2} u_\varepsilon(t)|^2,$$

and

$$F_\varepsilon(t) \leq \frac{\varepsilon |u'_\varepsilon(t)|^2}{\mu} + |A^{1/2} u_\varepsilon(t)|^2 + \frac{\delta}{2\nu} |A^{1/2} u_\varepsilon(t)|^2 + \frac{\varepsilon |u'_\varepsilon(t)|^2}{2M_3} + \frac{1}{2} |A^{1/2} u_\varepsilon(t)|^2,$$

from which (3.23) follows with

$$k_4 := \min \left\{ \frac{1}{2M_3}, \frac{1}{2} \right\}, \quad k_5 := \max \left\{ \frac{1}{\mu} + \frac{1}{2M_3}, \frac{3}{2} + \frac{\delta}{2\nu} \right\}.$$

Differential inequality for E_ε The time-derivative of (3.20) is

$$E'_\varepsilon(t) = -\frac{1}{(1+t)^p} \frac{|u'_\varepsilon(t)|^2}{c_\varepsilon(t)} \left(2 + \varepsilon \frac{c'_\varepsilon(t)(1+t)^p}{c_\varepsilon(t)} \right).$$

From (2.24) and (2.26) we have that

$$\varepsilon \frac{|c'_\varepsilon(t)|(1+t)^p}{c_\varepsilon(t)} \leq \frac{M_4}{\mu} \varepsilon,$$

so that $E'_\varepsilon(t) \leq 0$ for every $t \geq 0$, provided that ε is small enough. This proves (3.24).

Differential inequality for F_ε The time-derivative of (3.21) is

$$\begin{aligned} F'_\varepsilon(t) &= -\frac{1}{(1+t)^p} \frac{|u'_\varepsilon(t)|^2}{c_\varepsilon(t)} \left(2 + \varepsilon \frac{c'_\varepsilon(t)(1+t)^p}{c_\varepsilon(t)} - \varepsilon \delta c_\varepsilon(t) \right) \\ &\quad - \frac{\delta c_\varepsilon(t)}{(1+t)^p} |A^{1/2} u_\varepsilon(t)|^2 - \delta p \frac{|u_\varepsilon(t)|^2}{(1+t)^{2p+1}} - \frac{\varepsilon \delta p}{(1+t)^{1+p}} \langle u'_\varepsilon(t), u_\varepsilon(t) \rangle \end{aligned}$$

Therefore (3.25) holds true if and only if

$$\begin{aligned} &\frac{|u'_\varepsilon(t)|^2}{c_\varepsilon(t)} \left(2 + \varepsilon \frac{c'_\varepsilon(t)(1+t)^p}{c_\varepsilon(t)} - \varepsilon \delta c_\varepsilon(t) - \varepsilon \beta \right) + (\delta c_\varepsilon(t) - \beta) |A^{1/2} u_\varepsilon(t)|^2 + \\ &\left(\frac{\delta p}{(1+t)^{1+p}} - \frac{\delta \beta}{2} \frac{1}{(1+t)^{2p}} \right) |u_\varepsilon(t)|^2 + \left(\frac{\varepsilon \delta p}{1+t} - \frac{\varepsilon \delta \beta}{(1+t)^p} \right) \langle u'_\varepsilon(t), u_\varepsilon(t) \rangle \geq 0 \end{aligned} \quad (3.32)$$

holds true for every $t \geq T$, and every ε small enough.

Let S_1, \dots, S_4 denote the four terms in (3.32). Due to (2.24) through (2.26), for every small enough ε we have that

$$\varepsilon \frac{|c'_\varepsilon(t)|(1+t)^p}{c_\varepsilon(t)} \leq \frac{M_4}{\mu} \varepsilon \leq \frac{1}{3}, \quad \varepsilon \delta c_\varepsilon(t) \leq \varepsilon \delta M_3 \leq \frac{1}{3}, \quad \varepsilon \beta \leq \frac{1}{3},$$

hence

$$S_1 \geq \frac{|u'_\varepsilon(t)|^2}{c_\varepsilon(t)} \geq \frac{1}{M_3} |u'_\varepsilon(t)|^2. \quad (3.33)$$

Since $\delta\mu \geq \beta$, from (1.6) we have that

$$\begin{aligned} S_2 + S_3 &\geq (\delta\mu - \beta) |A^{1/2} u_\varepsilon(t)|^2 - \frac{\delta\beta}{2} \frac{1}{(1+t)^{2p}} |u_\varepsilon(t)|^2 \\ &\geq \left[(\delta\mu - \beta)\nu - \frac{\delta\beta}{2} \frac{1}{(1+T)^{2p}} \right] |u_\varepsilon(t)|^2 \end{aligned}$$

for every $t \geq T$. Due to the choices (3.18) and (3.19), in both cases the term in brackets is greater than or equal to ν , hence $S_2 + S_3 \geq \nu |u_\varepsilon(t)|^2$ for every $t \geq T$. Adding this inequality to (3.33), and applying the inequality between arithmetic and geometric mean, we deduce that

$$S_1 + S_2 + S_3 \geq \frac{1}{M_3} |u'_\varepsilon(t)|^2 + \nu |u_\varepsilon(t)|^2 \geq 2\sqrt{\frac{\nu}{M_3}} \cdot |u'_\varepsilon(t)| \cdot |u_\varepsilon(t)|.$$

As a consequence, if ε is small enough and $t \geq T$, we have that

$$S_1 + S_2 + S_3 \geq \varepsilon\delta(1+\beta) |u'_\varepsilon(t)| \cdot |u_\varepsilon(t)| \geq \left(\frac{\varepsilon\delta p}{1+t} + \frac{\varepsilon\delta\beta}{(1+t)^p} \right) |u'_\varepsilon(t)| \cdot |u_\varepsilon(t)| \geq |S_4|,$$

which proves (3.32), hence also (3.25).

Differential inequality for G_ε The time-derivative of (3.29) is

$$G'_\varepsilon(t) = -\frac{2}{\varepsilon} \frac{1}{(1+t)^p} |u'_\varepsilon(t)|^2 - \frac{2}{\varepsilon} c_\varepsilon(t) \langle Au_\varepsilon(t), u'_\varepsilon(t) \rangle.$$

From (2.25) we have that

$$-2c_\varepsilon(t) \langle Au_\varepsilon(t), u'_\varepsilon(t) \rangle \leq 2M_3 |u'_\varepsilon(t)| \cdot |Au_\varepsilon(t)| \leq \frac{|u'_\varepsilon(t)|^2}{(1+t)^p} + M_3^2 (1+t)^p |Au_\varepsilon(t)|^2,$$

hence

$$G'_\varepsilon(t) \leq -\frac{1}{\varepsilon} \frac{1}{(1+t)^p} G_\varepsilon(t) + \frac{M_3^2}{\varepsilon} (1+t)^p |Au_\varepsilon(t)|^2.$$

At this point (3.30) follows from (3.28).

The proof of Theorem 2.9 is thus complete. \square

3.4 Singular perturbation: preliminary estimates

In this subsection we begin the analysis of the singular perturbation problem in the linear setting. If we set

$$g_\varepsilon(t) := -(c_\varepsilon(t) - c(t))Au(t) - \varepsilon u''(t), \quad (3.34)$$

we have that $r_\varepsilon(t)$ and $\rho_\varepsilon(t)$ satisfy

$$\varepsilon r_\varepsilon''(t) + \frac{1}{(1+t)^p} r_\varepsilon'(t) + c_\varepsilon(t) A \rho_\varepsilon(t) = g_\varepsilon(t), \quad (3.35)$$

and

$$\rho_\varepsilon(0) = 0, \quad r_\varepsilon'(0) = 0.$$

In the next two results we prove estimates on $g_\varepsilon(t)$ and on the corrector $\theta_\varepsilon(t)$.

Lemma 3.5 *Let us consider the same assumptions of Theorem 2.10. Let $g_\varepsilon(t)$ be defined according to (3.34). Let $\Phi_{\beta,p}(t)$ be the solution of the Cauchy problem (3.8), (3.9), with $\beta > 0$ if $p > 0$, and $0 < \beta < 2\mu\nu$ if $p = 0$.*

Then we have the following estimates.

(1) *If $u_0 \in D(A^{3/2})$, then there exists a constant C such that*

$$\int_0^{+\infty} \frac{(1+t)^p}{\Phi_{\beta,p}(t)} \cdot |g_\varepsilon(t)|^2 dt \leq C\varepsilon^2 \quad \forall \varepsilon \in (0, \varepsilon_0). \quad (3.36)$$

(2) *If in addition we have that $u_0 \in D(A^2)$, then there exists a constant C such that*

$$\int_0^{+\infty} \frac{(1+t)^p}{\Phi_{\beta,p}(t)} \cdot |A^{1/2} g_\varepsilon(t)|^2 dt \leq C\varepsilon^2 \quad \forall \varepsilon \in (0, \varepsilon_0), \quad (3.37)$$

$$|g_\varepsilon(t)|^2 \leq C\varepsilon^2 \Phi_{\beta,p}(t) \quad \forall t \geq 0, \quad \forall \varepsilon \in (0, \varepsilon_0). \quad (3.38)$$

Proof From (3.34) and (2.27) we have that

$$|g_\varepsilon(t)|^2 \leq k_1 \varepsilon^2 |Au(t)|^2 + 2\varepsilon^2 |u''(t)|^2.$$

We can estimate $|Au(t)|^2$ and $|u''(t)|^2$, or their integrals, by means of Proposition 3.1. To this end, let us consider the function $\Psi_{\alpha,p}(t)$ defined in (3.1). We claim that, for every admissible value of p and β , there exists $\alpha > 0$ for which Proposition 3.1 applies, and such that

$$\frac{(1+t)^p}{\Phi_{\beta,p}(t)} \leq \frac{k_2}{\Psi_{\alpha,p}(t)} \quad \forall t \geq 0. \quad (3.39)$$

Indeed it is enough to take $\alpha = \beta$ if $p = 0$ (in which case there is basically nothing to prove), and any $\alpha \in (0, \gamma)$ if $p > 0$ (because in this case $\Psi_{\alpha,p}(t)$ has an exponential decay rate which is faster than the decay rate of $\Phi_{\beta,p}(t)$). Thus we have that

$$\int_0^{+\infty} \frac{(1+t)^p}{\Phi_{\beta,p}(t)} \cdot |g_\varepsilon(t)|^2 dt \leq k_3 \varepsilon^2 \left(\int_0^{+\infty} \frac{|Au(t)|^2}{\Psi_{\alpha,p}(t)} dt + \int_0^{+\infty} \frac{|u''(t)|^2}{\Psi_{\alpha,p}(t)} dt \right),$$

so that (3.36) follows from (3.3) with $k = 1$, and (3.4).

The proof of (3.37) is analogous: we just exploit (3.3) with $k = 2$, and (3.6) instead of (3.4).

It remains to prove (3.38). Let γ be the constant defined in (3.1). Then, in analogy with (3.39), we have that

$$\frac{(1+t)^{4p}}{\Phi_{\beta,p}(t)} \leq \frac{k_4}{\Psi_{\gamma,p}(t)} \quad \forall t \geq 0,$$

hence

$$\begin{aligned} \frac{|g_\varepsilon(t)|^2}{\Phi_{\beta,p}(t)} &= \frac{(1+t)^{4p}}{\Phi_{\beta,p}(t)} \cdot |g_\varepsilon(t)|^2 \cdot \frac{1}{(1+t)^{4p}} \leq k_4 \frac{|g_\varepsilon(t)|^2}{\Psi_{\gamma,p}(t)} \cdot \frac{1}{(1+t)^{4p}} \\ &\leq k_5 \varepsilon^2 \frac{|Au(t)|^2}{\Psi_{\gamma,p}(t)} + k_6 \varepsilon^2 \frac{|u''(t)|^2}{\Psi_{\gamma,p}(t)} \cdot \frac{1}{(1+t)^{4p}}. \end{aligned}$$

At this point (3.38) follows from (3.2) with $k = 2$, and (3.5). \square

Lemma 3.6 *Let us consider the same assumptions of Theorem 2.10. Let $\theta_\varepsilon(t)$ be the solution of the Cauchy problem (2.11), (2.28). Let $\Phi_{\beta,p}(t)$ be the solution of the Cauchy problem (3.8), (3.9).*

Let us assume that $4\varepsilon_0 \leq 1$, $2\varepsilon_0\beta \leq 1$, and that $\theta_0 \in D(A^{(k+1)/2})$ for some $k \in \mathbb{N}$.

Then there exists a constant C such that for every $\varepsilon \in (0, \varepsilon_0)$ we have that

$$\int_0^{+\infty} \frac{1}{\Phi_{\beta,p}(t)} \cdot (|A^{k/2}\theta'_\varepsilon(t)| + |A^{k/2}\theta'_\varepsilon(t)|^2 + |A^{(k+1)/2}\theta'_\varepsilon(t)|) dt \leq C\varepsilon. \quad (3.40)$$

Proof Let $z_\varepsilon(t)$ be the solution of equation

$$\varepsilon z'_\varepsilon(t) + \frac{1}{(1+t)^p} z_\varepsilon(t) = 0 \quad \forall t \geq 0, \quad (3.41)$$

with initial condition $z_\varepsilon(0) = 1$. It is easy to see that $\theta'_\varepsilon(t) = \theta_0 z_\varepsilon(t)$.

Since $0 \leq z_\varepsilon(t) \leq 1$ for every $t \geq 0$, we have also that $z_\varepsilon^2(t) \leq z_\varepsilon(t)$. Therefore, (3.40) is proved if we show that

$$\int_0^{+\infty} \frac{z_\varepsilon(t)}{\Phi_{\beta,p}(t)} dt \leq 4\varepsilon. \quad (3.42)$$

Let us set $w_\varepsilon(t) := z_\varepsilon(t) \cdot [\Phi_{\beta,p}(t)]^{-1}$. From (3.41) and (3.8), it turns out that $w_\varepsilon(t)$ is the solution of the ordinary differential equation

$$w'_\varepsilon(t) = - \left(\frac{1}{\varepsilon} - \beta \right) \frac{1}{(1+t)^p} w_\varepsilon(t) \quad \forall t \geq 0, \quad (3.43)$$

with initial datum $w_\varepsilon(0) = 1$. On the other hand, when $2\varepsilon\beta \leq 1$, it is easy to show that $y_\varepsilon(t) := (1+t)^{-1/(2\varepsilon)}$ is a supersolution of (3.43). Indeed we have that

$$y'_\varepsilon(t) = - \frac{1}{2\varepsilon} \frac{y_\varepsilon(t)}{1+t} \geq - \frac{1}{2\varepsilon} \frac{y_\varepsilon(t)}{(1+t)^p} \geq - \left(\frac{1}{\varepsilon} - \beta \right) \frac{y_\varepsilon(t)}{(1+t)^p}.$$

Since $y_\varepsilon(0) = w_\varepsilon(0)$, the standard comparison principle gives that $w_\varepsilon(t) \leq y_\varepsilon(t)$ for every $t \geq 0$. Since $4\varepsilon \leq 1$, it follows that

$$\int_0^{+\infty} w_\varepsilon(t) dt \leq \int_0^{+\infty} \frac{1}{(1+t)^{1/(2\varepsilon)}} dt = \frac{2\varepsilon}{1-2\varepsilon} \leq 4\varepsilon.$$

This completes the proof of (3.42), hence also the proof of (3.40). \square

3.5 Proof of Theorem 2.10

Let us describe the strategy of the proof, which is similar to Theorem 2.9. Let us take any admissible value β , which means any $\beta \in (0, 2\mu\nu)$ if $p = 0$, and any $\beta > 0$ if $p > 0$. Let $\Phi_{\beta,p}(t)$ be the solution of the Cauchy problem (3.8), (3.9).

The conclusions of statement (1) of Theorem 2.10 are equivalent to showing that

$$\Gamma_{r,\varepsilon}(t) \leq k_1 \Phi_{\beta,p}(t) \quad \forall t \geq 0 \quad (3.44)$$

for the admissible values of β .

Let μ be the constant in (2.24), and let us choose δ, σ, T in such a way that

$$\delta := \frac{4(\beta+1)\nu}{2\mu\nu - \beta}, \quad \sigma := \mu\nu - \frac{\beta}{2}, \quad T := 0 \quad (3.45)$$

if $p = 0$ (note that $\delta > 0$), and

$$\delta := \frac{\beta+2}{\mu}, \quad \sigma := 1, \quad (1+T)^{2p} \geq \frac{\delta}{2\nu}(\beta+\sigma) \quad (3.46)$$

if $p > 0$.

For every $\varepsilon \in (0, \varepsilon_0)$, we consider the energies

$$\mathcal{E}_\varepsilon(t) := \frac{\varepsilon |r'_\varepsilon(t)|^2}{c_\varepsilon(t)} + |A^{1/2} \rho_\varepsilon(t)|^2, \quad (3.47)$$

$$\mathcal{F}_\varepsilon(t) := \frac{\varepsilon |r'_\varepsilon(t)|^2}{c_\varepsilon(t)} + |A^{1/2} \rho_\varepsilon(t)|^2 + \frac{\varepsilon \delta}{(1+t)^p} \langle r'_\varepsilon(t), \rho_\varepsilon(t) \rangle + \frac{\delta}{2} \frac{1}{(1+t)^{2p}} |\rho_\varepsilon(t)|^2. \quad (3.48)$$

The arguments used in the proof of (3.22) and (3.23) can be adapted word-by-word to the energies $\mathcal{E}_\varepsilon(t)$ and $\mathcal{F}_\varepsilon(t)$. We obtain that there exist positive constants k_2, \dots, k_5 such that

$$k_2 (\varepsilon |r'_\varepsilon(t)|^2 + |A^{1/2} \rho_\varepsilon(t)|^2) \leq \mathcal{E}_\varepsilon(t) \leq k_3 (\varepsilon |r'_\varepsilon(t)|^2 + |A^{1/2} \rho_\varepsilon(t)|^2), \quad (3.49)$$

$$k_4 (\varepsilon |r'_\varepsilon(t)|^2 + |A^{1/2} \rho_\varepsilon(t)|^2) \leq \mathcal{F}_\varepsilon(t) \leq k_5 (\varepsilon |r'_\varepsilon(t)|^2 + |A^{1/2} \rho_\varepsilon(t)|^2) \quad (3.50)$$

for every $t \geq 0$, provided that ε is small enough.

Moreover, we claim that there exists $\varepsilon_2 \in (0, \varepsilon_0)$ such that

$$\mathcal{E}'_\varepsilon(t) \leq \psi_{1,\varepsilon}(t) \sqrt{\mathcal{E}_\varepsilon(t)} + \psi_{2,\varepsilon}(t) \quad \forall t \geq 0, \quad \forall \varepsilon \in (0, \varepsilon_2), \quad (3.51)$$

$$\mathcal{F}'_\varepsilon(t) \leq -\frac{\beta}{(1+t)^p} \mathcal{F}_\varepsilon(t) + \psi_{3,\varepsilon}(t) \quad \forall t \geq T, \quad \forall \varepsilon \in (0, \varepsilon_2), \quad (3.52)$$

where the functions $\psi_{i,\varepsilon}(t)$ (with $i = 1, 2, 3$) are nonnegative continuous functions depending on $A^{1/2} \theta'_\varepsilon(t)$ and $g_\varepsilon(t)$, and such that

$$\int_0^{+\infty} \psi_{1,\varepsilon}(t) dt \leq k_6 \varepsilon, \quad \int_0^{+\infty} \psi_{2,\varepsilon}(t) dt \leq k_7 \varepsilon^2, \quad (3.53)$$

$$\int_0^{+\infty} \frac{\psi_{3,\varepsilon}(t)}{\Phi_{\beta,p}(t)} dt \leq k_8 \varepsilon^2. \quad (3.54)$$

Let us assume that we have proved these claims. Thanks to (3.51) and (3.53), we can apply Lemma 3.3 to the function $\mathcal{E}_\varepsilon(t)$ (note that now $\mathcal{E}_\varepsilon(0) = 0$). We obtain that

$$\mathcal{E}_\varepsilon(t) \leq k_9 \varepsilon^2 \quad \forall t \geq 0. \quad (3.55)$$

Due to the estimate from below in (3.49), this implies that

$$\varepsilon |r'_\varepsilon(t)|^2 + |A^{1/2} \rho_\varepsilon(t)|^2 \leq \frac{1}{k_2} \mathcal{E}_\varepsilon(t) \leq k_{10} \varepsilon^2$$

for every $t \geq 0$. Since $\Phi_{\beta,p}(t)$ is decreasing, we can conclude that

$$\varepsilon |r'_\varepsilon(t)|^2 + |A^{1/2} \rho_\varepsilon(t)|^2 \leq \frac{k_{10} \varepsilon^2}{\Phi_{\beta,p}(T)} \cdot \Phi_{\beta,p}(t) = k_{11} \varepsilon^2 \Phi_{\beta,p}(t) \quad \forall t \in [0, T]. \quad (3.56)$$

For $t \geq T$, we exploit (3.52). First of all, from (3.56) with $t = T$, and the estimate from above in (3.50), we have that

$$\mathcal{F}_\varepsilon(T) \leq k_5 (\varepsilon |r'_\varepsilon(T)|^2 + |A^{1/2} \rho_\varepsilon(T)|^2) \leq k_{12} \varepsilon^2 \Phi_{\beta,p}(T).$$

Due to (3.52) and (3.54), we can apply Lemma 3.4 to the function $\mathcal{F}_\varepsilon(t)$. We obtain that $\mathcal{F}_\varepsilon(t) \leq k_{13}\varepsilon^2\Phi_{\beta,p}(t)$ for every $t \geq T$. Exploiting this inequality, the estimate from below in (3.50), and (3.56), we conclude that

$$\varepsilon|r'_\varepsilon(t)|^2 + |A^{1/2}\rho_\varepsilon(t)|^2 \leq k_{14}\varepsilon^2\Phi_{\beta,p}(t) \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_2).$$

Since the operator is coercive, this estimate on $|A^{1/2}\rho_\varepsilon(t)|^2$ yields an analogous estimate on $|\rho_\varepsilon(t)|^2$. This completes the proof of (3.44), hence of statement (1), for initial data $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$, the regularity of data being required in the verification of (3.53) and (3.54).

Let us proceed now to statement (2), where it is assumed that $(u_0, u_1) \in D(A^2) \times D(A)$, and it is required to prove in addition that

$$\varepsilon|A^{1/2}r'_\varepsilon(t)|^2 + |A\rho_\varepsilon(t)|^2 + |r'_\varepsilon(t)|^2 \leq k_{15}\varepsilon^2\Phi_{\beta,p}(t) \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_1)$$

for some $\varepsilon_1 \in (0, \varepsilon_2]$. Due to the linearity of (3.35), an analogous identity holds true with $A^{1/2}\rho_\varepsilon(t)$, $A^{1/2}r_\varepsilon(t)$, and $A^{1/2}g_\varepsilon(t)$ instead of $\rho_\varepsilon(t)$, $r_\varepsilon(t)$, and $g_\varepsilon(t)$, respectively. So we can repeat the arguments used sofar, paying attention to verifying (3.53) and (3.54) also for the new functions $\psi_{\varepsilon,i}(t)$, which now depend on $A\theta'_\varepsilon(t)$ and $A^{1/2}g_\varepsilon(t)$. We end up with

$$\varepsilon|A^{1/2}r'_\varepsilon(t)|^2 + |A\rho_\varepsilon(t)|^2 \leq k_{16}\varepsilon^2\Phi_{\beta,p}(t) \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_2). \quad (3.57)$$

It remains to prove the ε -independent estimates on $r'_\varepsilon(t)$. To this end, we set

$$\mathcal{G}_\varepsilon(t) := |r'_\varepsilon(t)|^2, \quad (3.58)$$

and we claim that

$$\mathcal{G}'_\varepsilon(t) \leq -\frac{1}{\varepsilon} \frac{1}{(1+t)^p} \mathcal{G}_\varepsilon(t) + \frac{1}{\varepsilon} (1+t)^p \cdot k_{17}\varepsilon^2\Phi_{\beta,p}(t) \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_2). \quad (3.59)$$

If we prove the claim, then from Lemma 3.2 it follows that (note that now $\mathcal{G}_\varepsilon(0) = 0$)

$$|r'_\varepsilon(t)|^2 = \mathcal{G}_\varepsilon(t) \leq k_{18}\varepsilon^2(1+t)^{2p}\Phi_{\beta,p}(t) \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_2).$$

Finally, when $p > 0$, we can get free of the factor $(1+t)^{2p}$ exactly as in the proof of Theorem 2.9, possibly changing ε_2 with some smaller ε_1 .

So we are left to proving (3.51) through (3.54), both in the case of initial data $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$, and in the case $(u_0, u_1) \in D(A^2) \times D(A)$, and (3.59) in the second case.

Differential inequality for \mathcal{E}_ε The time-derivative of (3.47) is

$$\begin{aligned}\mathcal{E}'_\varepsilon(t) &= -\frac{1}{(1+t)^p} \frac{|r'_\varepsilon(t)|^2}{c_\varepsilon(t)} \left(2 + \varepsilon \frac{c'_\varepsilon(t)(1+t)^p}{c_\varepsilon(t)} \right) \\ &\quad + \frac{2}{c_\varepsilon(t)} \langle r'_\varepsilon(t), g_\varepsilon(t) \rangle + 2 \langle A\rho_\varepsilon(t), \theta'_\varepsilon(t) \rangle.\end{aligned}\tag{3.60}$$

By standard inequalities we have that

$$\begin{aligned}2 \langle A\rho_\varepsilon(t), \theta'_\varepsilon(t) \rangle &\leq 2|A^{1/2}\theta'_\varepsilon(t)| \cdot |A^{1/2}\rho_\varepsilon(t)| \leq 2|A^{1/2}\theta'_\varepsilon(t)|\sqrt{\mathcal{E}_\varepsilon(t)}, \\ \frac{2}{c_\varepsilon(t)} \langle r'_\varepsilon(t), g_\varepsilon(t) \rangle &\leq \frac{1}{(1+t)^p} \frac{|r'_\varepsilon(t)|^2}{c_\varepsilon(t)} + \frac{1}{c_\varepsilon(t)} (1+t)^p |g_\varepsilon(t)|^2.\end{aligned}$$

Plugging these estimates into (3.60), when ε is small enough we obtain that

$$\mathcal{E}'_\varepsilon(t) \leq 2|A^{1/2}\theta'_\varepsilon(t)|\sqrt{\mathcal{E}_\varepsilon(t)} + \frac{1}{\mu} (1+t)^p |g_\varepsilon(t)|^2,$$

which is exactly (3.51) with

$$\psi_{1,\varepsilon}(t) := 2|A^{1/2}\theta'_\varepsilon(t)|, \quad \psi_{2,\varepsilon}(t) := \frac{1}{\mu} (1+t)^p |g_\varepsilon(t)|^2.$$

When $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$, we have that $\theta_0 \in D(A^{1/2})$, hence (3.53) follows from (3.40) with $k = 0$, and (3.36).

When $(u_0, u_1) \in D(A^2) \times D(A)$, we have that $\theta_0 \in D(A)$, and we need (3.53) with $\psi_{1,\varepsilon}(t) := 2|A\theta'_\varepsilon(t)|$, and $\psi_{2,\varepsilon}(t) := \mu^{-1}(1+t)^p |A^{1/2}g_\varepsilon(t)|^2$. Due to the regularity of θ_0 , estimate (3.53) follows in this case from (3.40) with $k = 1$, and (3.37).

Differential inequality for \mathcal{F}_ε The time-derivative of (3.48) is

$$\begin{aligned}\mathcal{F}'_\varepsilon(t) &= -\frac{1}{(1+t)^p} \frac{|r'_\varepsilon(t)|^2}{c_\varepsilon(t)} \left(2 + \varepsilon \frac{c'_\varepsilon(t)(1+t)^p}{c_\varepsilon(t)} - \varepsilon \delta c_\varepsilon(t) \right) \\ &\quad - \frac{\delta c_\varepsilon(t)}{(1+t)^p} |A^{1/2}\rho_\varepsilon(t)|^2 - \delta p \frac{|\rho_\varepsilon(t)|^2}{(1+t)^{2p+1}} - \frac{\varepsilon \delta p}{(1+t)^{1+p}} \langle r'_\varepsilon(t), \rho_\varepsilon(t) \rangle \\ &\quad + \frac{\varepsilon \delta}{(1+t)^p} \langle r'_\varepsilon(t), \theta'_\varepsilon(t) \rangle + 2 \langle A^{1/2}\rho_\varepsilon(t), A^{1/2}\theta'_\varepsilon(t) \rangle + \frac{\delta}{(1+t)^{2p}} \langle \rho_\varepsilon(t), \theta'_\varepsilon(t) \rangle \\ &\quad + \frac{2}{c_\varepsilon(t)} \langle r'_\varepsilon(t), g_\varepsilon(t) \rangle + \frac{\delta}{(1+t)^p} \langle \rho_\varepsilon(t), g_\varepsilon(t) \rangle \\ &= I_1 + \dots + I_9.\end{aligned}\tag{3.61}$$

Let us estimate some of the terms. Clearly we have that $I_3 \leq 0$. From (3.55) we have that

$$I_6 \leq 2 |A^{1/2}\rho_\varepsilon(t)| \cdot |A^{1/2}\theta'_\varepsilon(t)| \leq k_{19}\varepsilon |A^{1/2}\theta'_\varepsilon(t)|,$$

$$I_7 \leq \frac{\delta}{(1+t)^{2p}} |\rho_\varepsilon(t)| \cdot |\theta'_\varepsilon(t)| \leq \frac{\delta}{(1+t)^{2p}} \frac{1}{\sqrt{\nu}} |A^{1/2} \rho_\varepsilon(t)| \cdot |\theta'_\varepsilon(t)| \leq k_{20} \varepsilon |\theta'_\varepsilon(t)|.$$

From standard inequalities we have that

$$I_5 \leq \frac{\varepsilon \delta}{(1+t)^p} |r'_\varepsilon(t)| \cdot |\theta'_\varepsilon(t)| \leq \frac{\varepsilon \delta}{2} \frac{1}{(1+t)^p} \frac{|r'_\varepsilon(t)|^2}{c_\varepsilon(t)} + \frac{\varepsilon \delta}{2} \frac{c_\varepsilon(t)}{(1+t)^p} |\theta'_\varepsilon(t)|^2,$$

$$I_8 \leq \frac{2}{c_\varepsilon(t)} |r'_\varepsilon(t)| \cdot |g_\varepsilon(t)| \leq \frac{1}{2} \frac{1}{(1+t)^p} \frac{|r'_\varepsilon(t)|^2}{c_\varepsilon(t)} + \frac{2}{c_\varepsilon(t)} (1+t)^p |g_\varepsilon(t)|^2,$$

$$I_9 \leq \frac{\delta}{(1+t)^p} |\rho_\varepsilon(t)| \cdot |g_\varepsilon(t)| \leq \frac{\delta \sigma}{2} \frac{1}{(1+t)^{3p}} |\rho_\varepsilon(t)|^2 + \frac{\delta}{2\sigma} (1+t)^p |g_\varepsilon(t)|^2.$$

Plugging all these estimates into (3.61), and recalling once more assumptions (2.24) through (2.26), we obtain that

$$\begin{aligned} \mathcal{F}'_\varepsilon(t) &\leq -\frac{1}{(1+t)^p} \frac{|r'_\varepsilon(t)|^2}{c_\varepsilon(t)} \left(\frac{3}{2} + \varepsilon \frac{c'_\varepsilon(t)(1+t)^p}{c_\varepsilon(t)} - \varepsilon \delta c_\varepsilon(t) - \frac{\varepsilon \delta}{2} \right) \\ &\quad - \frac{\delta c_\varepsilon(t)}{(1+t)^p} |A^{1/2} \rho_\varepsilon(t)|^2 + \frac{\delta \sigma}{2} \frac{1}{(1+t)^{3p}} |\rho_\varepsilon(t)|^2 - \frac{\varepsilon \delta p}{(1+t)^{1+p}} \langle r'_\varepsilon(t), \rho_\varepsilon(t) \rangle \\ &\quad + k_{21} \varepsilon (|\theta'_\varepsilon(t)| + |\theta'_\varepsilon(t)|^2 + |A^{1/2} \theta'_\varepsilon(t)|) + k_{22} (1+t)^p |g_\varepsilon(t)|^2. \end{aligned} \quad (3.62)$$

Let $\psi_{3,\varepsilon}(t)$ denote the sum of the two terms of the last line. Then (3.52) is proved if we show that the sum of the terms in the first two lines is less than or equal to $-\beta(1+t)^{-p} \mathcal{F}_\varepsilon(t)$ for every $t \geq T$. In turn, this is equivalent to showing that

$$\begin{aligned} &\frac{|r'_\varepsilon(t)|^2}{c_\varepsilon(t)} \left(\frac{3}{2} + \varepsilon \frac{c'_\varepsilon(t)(1+t)^p}{c_\varepsilon(t)} - \varepsilon \delta c_\varepsilon(t) - \frac{\varepsilon \delta}{2} - \varepsilon \beta \right) + (\delta c_\varepsilon(t) - \beta) |A^{1/2} \rho_\varepsilon(t)|^2 + \\ &\quad - \frac{\delta(\sigma + \beta)}{2} \frac{|\rho_\varepsilon(t)|^2}{(1+t)^{2p}} + \left(\frac{\varepsilon \delta p}{1+t} - \frac{\varepsilon \delta \beta}{(1+t)^p} \right) \langle r'_\varepsilon(t), \rho_\varepsilon(t) \rangle \geq 0 \end{aligned} \quad (3.63)$$

holds true for every $t \geq T$.

Let S_1, \dots, S_4 denote the four terms in (3.63), which we estimate as in the proof of Theorem 2.9. From the smallness of ε we have that

$$S_1 \geq \frac{|r'_\varepsilon(t)|^2}{c_\varepsilon(t)} \geq \frac{1}{M_3} |r'_\varepsilon(t)|^2. \quad (3.64)$$

Since $\delta \mu \geq \beta$, from (1.6) we have that

$$\begin{aligned} S_2 + S_3 &\geq (\delta \mu - \beta) |A^{1/2} \rho_\varepsilon(t)|^2 - \frac{\delta(\sigma + \beta)}{2} \frac{1}{(1+t)^{2p}} |\rho_\varepsilon(t)|^2 \\ &\geq \left[(\delta \mu - \beta) \nu - \frac{\delta(\sigma + \beta)}{2} \frac{1}{(1+T)^{2p}} \right] |\rho_\varepsilon(t)|^2 \end{aligned}$$

for every $t \geq T$. Due to the choices (3.45) and (3.46), in both cases the term in brackets is greater than or equal to ν , hence $S_2 + S_3 \geq \nu|\rho_\varepsilon(t)|^2$ for every $t \geq T$. Now we add this inequality to (3.64), and we apply the inequality between arithmetic and geometric mean, exactly as in the proof of Theorem 2.9. If ε is small enough we obtain that

$$\begin{aligned} S_1 + S_2 + S_3 &\geq \frac{1}{M_3}|r'_\varepsilon(t)|^2 + \nu|\rho_\varepsilon(t)|^2 \geq 2\sqrt{\frac{\nu}{M_3}} \cdot |r'_\varepsilon(t)| \cdot |\rho_\varepsilon(t)| \\ &\geq \varepsilon\delta(1+\beta)|r'_\varepsilon(t)| \cdot |\rho_\varepsilon(t)| \geq \left(\frac{\varepsilon\delta p}{1+t} + \frac{\varepsilon\delta\beta}{(1+t)^p}\right) |r'_\varepsilon(t)| \cdot |\rho_\varepsilon(t)| \geq |S_4|, \end{aligned}$$

which proves (3.63), hence also (3.52).

It remains to prove (3.54), with $\psi_{3,\varepsilon}(t)$ equal to the sum of the two terms in the last line of (3.62).

When $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$, we have that $\theta_0 \in D(A^{1/2})$, hence (3.54) follows from (3.40) with $k = 0$, and (3.36).

When $(u_0, u_1) \in D(A^2) \times D(A)$, we have that $\theta_0 \in D(A)$, and we need (3.54) with

$$\psi_{3,\varepsilon}(t) := k_{23}\varepsilon (|A^{1/2}\theta'_\varepsilon(t)| + |A^{1/2}\theta'_\varepsilon(t)|^2 + |A\theta'_\varepsilon(t)|) + k_{24}(1+t)^p|A^{1/2}g_\varepsilon(t)|^2.$$

Due to the regularity of θ_0 , estimate (3.54) follows in this case from (3.40) with $k = 1$, and (3.37).

Differential inequality for \mathcal{G}_ε The time-derivative of (3.58) is

$$\mathcal{G}'_\varepsilon(t) = -\frac{2}{\varepsilon}\frac{1}{(1+t)^p}|r'_\varepsilon(t)|^2 - \frac{2}{\varepsilon}c_\varepsilon(t)\langle A\rho_\varepsilon(t), r'_\varepsilon(t) \rangle + \frac{2}{\varepsilon}\langle g_\varepsilon(t), r'_\varepsilon(t) \rangle.$$

From standard inequalities we have that

$$\begin{aligned} -\frac{2}{\varepsilon}c_\varepsilon(t)\langle A\rho_\varepsilon(t), r'_\varepsilon(t) \rangle &\leq \frac{1}{2\varepsilon}\frac{1}{(1+t)^p}|r'_\varepsilon(t)|^2 + \frac{k_{25}}{\varepsilon}(1+t)^p|A\rho_\varepsilon(t)|^2, \\ \frac{2}{\varepsilon}\langle g_\varepsilon(t), r'_\varepsilon(t) \rangle &\leq \frac{1}{2\varepsilon}\frac{1}{(1+t)^p}|r'_\varepsilon(t)|^2 + \frac{2}{\varepsilon}(1+t)^p|g_\varepsilon(t)|^2, \end{aligned}$$

hence

$$\mathcal{G}'_\varepsilon(t) \leq -\frac{1}{\varepsilon}\frac{1}{(1+t)^p}|r'_\varepsilon(t)|^2 + \frac{k_{25}}{\varepsilon}(1+t)^p|A\rho_\varepsilon(t)|^2 + \frac{2}{\varepsilon}(1+t)^p|g_\varepsilon(t)|^2.$$

At this point (3.59) follows from (3.57) and (3.38). This completes the proof of Theorem 2.10. \square

3.6 Proof of Theorem 2.11

Let us set

$$H_\varepsilon(t) := \left(\varepsilon \frac{|u'_\varepsilon(t)|^2}{c_\varepsilon(t)} + |A^{1/2}u_\varepsilon(t)|^2 \right) \frac{1}{\Phi(t)} \quad \forall t \geq 0.$$

Due to (2.24) and (2.25), proving (2.16) is equivalent to showing that $H_\varepsilon(t) \rightarrow +\infty$ as $t \rightarrow +\infty$. Since $(u_0, u_1) \neq (0, 0)$, the solution is nontrivial in the sense that $H_\varepsilon(t) > 0$ for every $t \geq 0$. Moreover we have that

$$\begin{aligned} H'_\varepsilon(t) &= \frac{1}{(1+t)^p} \frac{1}{\Phi(t)} \frac{\varepsilon |u'_\varepsilon(t)|^2}{c_\varepsilon(t)} \left(-\frac{\Phi'(t)}{\Phi(t)} (1+t)^p - \frac{2}{\varepsilon} - \frac{c'_\varepsilon(t)(1+t)^p}{c_\varepsilon(t)} \right) \\ &\quad + \frac{1}{(1+t)^p} \frac{1}{\Phi(t)} |A^{1/2}u_\varepsilon(t)|^2 \left(-\frac{\Phi'(t)}{\Phi(t)} (1+t)^p \right). \end{aligned}$$

As usual, we have that

$$\frac{|c'_\varepsilon(t)|(1+t)^p}{c_\varepsilon(t)} \leq \frac{M_4}{\mu}.$$

Therefore, assumption (2.15) implies the existence of $T > 0$ (depending on ε , but this is not important) such that

$$-\frac{\Phi'(t)}{\Phi(t)} (1+t)^p - \frac{2}{\varepsilon} - \frac{c'_\varepsilon(t)(1+t)^p}{c_\varepsilon(t)} \geq 1 \quad \text{and} \quad -\frac{\Phi'(t)}{\Phi(t)} (1+t)^p \geq 1$$

for every $t \geq T$, hence

$$H'_\varepsilon(t) \geq \frac{1}{(1+t)^p} H_\varepsilon(t) \quad \forall t \geq T.$$

Since $H_\varepsilon(T) > 0$, and $p \leq 1$, this differential inequality implies that $H_\varepsilon(t) \rightarrow +\infty$ as $t \rightarrow +\infty$. \square

3.7 Proof of Theorems 2.1, 2.2, 2.3, 2.4

The existence of solutions to (1.3), (1.4), and (1.1), (1.2) follows from Theorem A. Let us set now

$$c(t) := m(|A^{1/2}u(t)|^2), \quad c_\varepsilon(t) := m(|A^{1/2}u_\varepsilon(t)|^2).$$

With a standard approximation procedure, we can assume that $m(\sigma)$ is of class C^1 , and not just locally Lipschitz continuous. As a consequence, also $c(t)$ and $c_\varepsilon(t)$ are of class C^1 . If we show that $c(t)$ and $c_\varepsilon(t)$ satisfy (2.21) through (2.27), then all conclusions of Theorems 2.1 through 2.4 follow from the corresponding conclusions of Theorems 2.8 through 2.11.

Assumptions (2.21) and (2.24) follow from (1.5).

Assumptions (2.22) and (2.25) follow from the fact that $|A^{1/2}u(t)|^2$ and $|A^{1/2}u_\varepsilon(t)|^2$ are bounded because of (2.2) and (2.4), respectively.

Since

$$c'(t) = 2m' (|A^{1/2}u(t)|^2) \langle Au(t), u'(t) \rangle,$$

assumption (2.23) follows from the boundedness of $|u'(t)|$, $|A^{1/2}u(t)|$, and $|Au(t)|$, resulting from (2.2).

Similarly, we have that

$$c'_\varepsilon(t) = 2m' (|A^{1/2}u_\varepsilon(t)|^2) \langle Au_\varepsilon(t), u'_\varepsilon(t) \rangle,$$

and therefore estimate (2.4) implies that

$$|c'_\varepsilon(t)| \leq k_1 |Au_\varepsilon(t)| \cdot |u'_\varepsilon(t)| \leq k_2 \frac{1}{(1+t)^{1+p}} \cdot \frac{1}{1+t} \leq \frac{k_2}{(1+t)^p},$$

which is exactly (2.26).

It remains to prove (2.27). To this end, we first remark that

$$\begin{aligned} \left| |A^{1/2}u_\varepsilon(t)|^2 - |A^{1/2}u(t)|^2 \right| &= \left| \langle A^{1/2}(u_\varepsilon(t) + u(t)), A^{1/2}(u_\varepsilon(t) - u(t)) \rangle \right| \\ &\leq (|A^{1/2}u_\varepsilon(t)| + |A^{1/2}u(t)|) \cdot |A^{1/2}\rho_\varepsilon(t)|. \end{aligned}$$

Now $|A^{1/2}u_\varepsilon(t)|$ and $|A^{1/2}u(t)|$ are bounded because of (2.2) and (2.4), and $|A^{1/2}\rho_\varepsilon(t)|$ can be estimated by means of (2.5). Since $m(\sigma)$ is (locally) Lipschitz continuous, we obtain that

$$|c_\varepsilon(t) - c(t)| \leq k_3 \left| |A^{1/2}u_\varepsilon(t)|^2 - |A^{1/2}u(t)|^2 \right| \leq k_4 \varepsilon,$$

which is exactly (2.27). \square

References

- [1] R. CHILL, A. HARAUX; An optimal estimate for the time singular limit of an abstract wave equation. *Funkcial. Ekvac.* **47** (2004), no. 2, 277–290.
- [2] M. GHISI; Hyperbolic-parabolic singular perturbation for mildly degenerate Kirchhoff equations with weak dissipation. *Adv. Differential Equations* **17** (2012), no. 1–2, 1–36.
- [3] M. GHISI, M. GOBBINO; Global-in-time uniform convergence for linear hyperbolic-parabolic singular perturbations. *Acta Math. Sin. (Engl. Ser.)* **22** (2006), no. 4, 1161–1170.
- [4] M. GHISI, M. GOBBINO; Hyperbolic-parabolic singular perturbation for mildly degenerate Kirchhoff equations: global-in-time error estimates. *Commun. Pure Appl. Anal.* **8** (2009), no. 4, 1313–1332.

- [5] M. GHISI, M. GOBBINO; Hyperbolic-parabolic singular perturbation for non-degenerate Kirchhoff equations with critical weak dissipation. *Math. Ann.* doi:10.1007/s00208-011-0765-x.
- [6] M. GHISI, M. GOBBINO; Hyperbolic-parabolic singular perturbation for Kirchhoff equations with weak dissipation. *Rend. Ist. Mat. Univ. Trieste* **42** Suppl. (2010), 67–88.
- [7] M. GHISI, M. GOBBINO; Hyperbolic-parabolic singular perturbation for mildly degenerate Kirchhoff equations: Decay-error estimates. *J. Differential Equations* (2012), doi:10.1016/j.jde.2012.02.019.
- [8] M. GHISI, M. GOBBINO; Optimal decay-error estimates for the hyperbolic-parabolic singular perturbation of a degenerate nonlinear equation. Preprint. [arXiv:1203.0865v1](https://arxiv.org/abs/1203.0865v1) [math.AP]
- [9] M. GOBBINO; Quasilinear degenerate parabolic equations of Kirchhoff type. *Math. Methods Appl. Sci.* **22** (1999), no. 5, 375–388.
- [10] H. HASHIMOTO, T. YAMAZAKI; Hyperbolic-parabolic singular perturbation for quasilinear equations of Kirchhoff type. *J. Differential Equations* **237** (2007), no. 2, 491–525.
- [11] J. L. LIONS; *Perturbations singulières dans les problèmes aux limites et en control optimal*, Lecture Notes in Mathematics, Vol. 323. Springer-Verlag, Berlin-New York, 1973.
- [12] J. WIRTH; Wave equations with time-dependent dissipation. II. Effective dissipation. *J. Differential Equations* **232** (2007), no. 1, 74–103.
- [13] J. WIRTH; Scattering and modified scattering for abstract wave equations with time-dependent dissipation. *Adv. Differential Equations* **12** (2007), no. 10, 1115–1133.
- [14] T. YAMAZAKI; Asymptotic behavior for abstract wave equations with decaying dissipation. *Adv. Differential Equations* **11** (2006), 419–456.
- [15] T. YAMAZAKI; Hyperbolic-parabolic singular perturbation for quasilinear equations of Kirchhoff type with weak dissipation. *Math. Methods Appl. Sci.* **32** (2009), no. 15, 1893–1918.
- [16] T. YAMAZAKI; Hyperbolic-parabolic singular perturbation for quasilinear equations of Kirchhoff type with weak dissipation of critical power. Preprint.